Network Flow Applications

Bicycle Sharing

Ever wonder why they don't all end up at the bottom of a hill somewhere?

Bicycle Sharing Problem

Let $d(v)$ be the number of additional bikes needed at location v for the following morning's commute

$d(v) > 0$: demand

$d(v) < 0$: supply

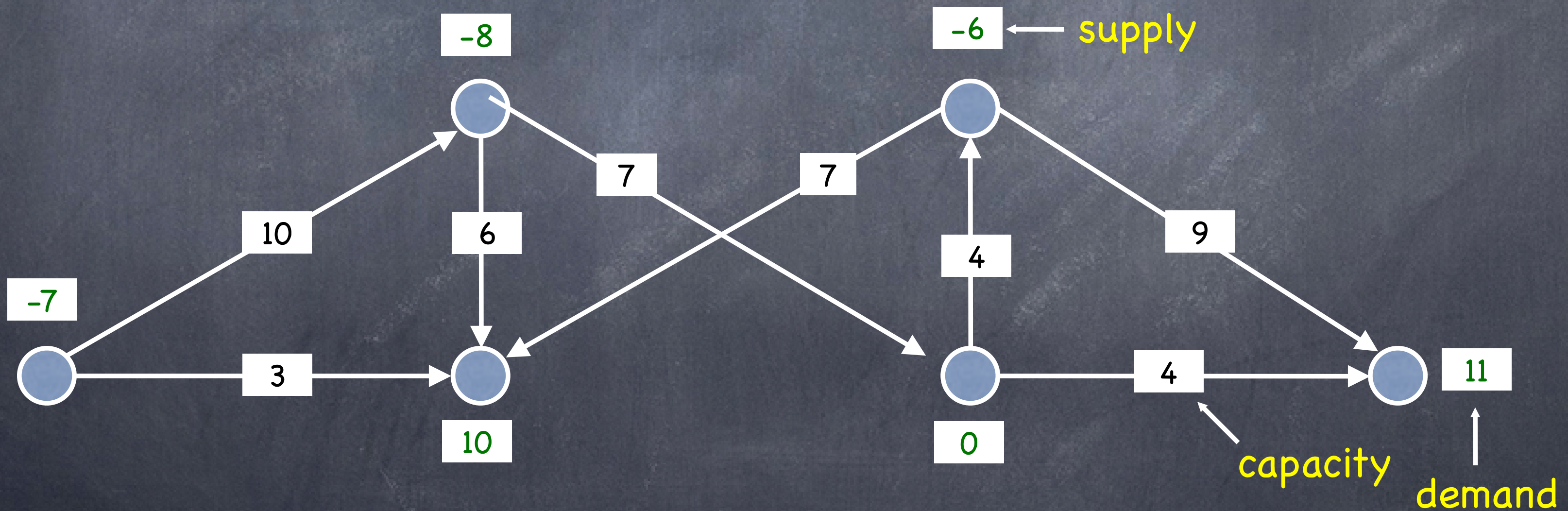
For certain location pairs $e = (u, v)$, a van is available to transport up to $c(e)$ bicycles from u to v

Do you have enough vans to move the bikes?

Circulation with Demands

- Directed graph G = (V, E).
- Edge capacities c(e), e ∈ E.
- Node supply/demand d(v), v ∈ V.

d(v) > 0 → "demand node"
d(v) < 0 → "supply node"

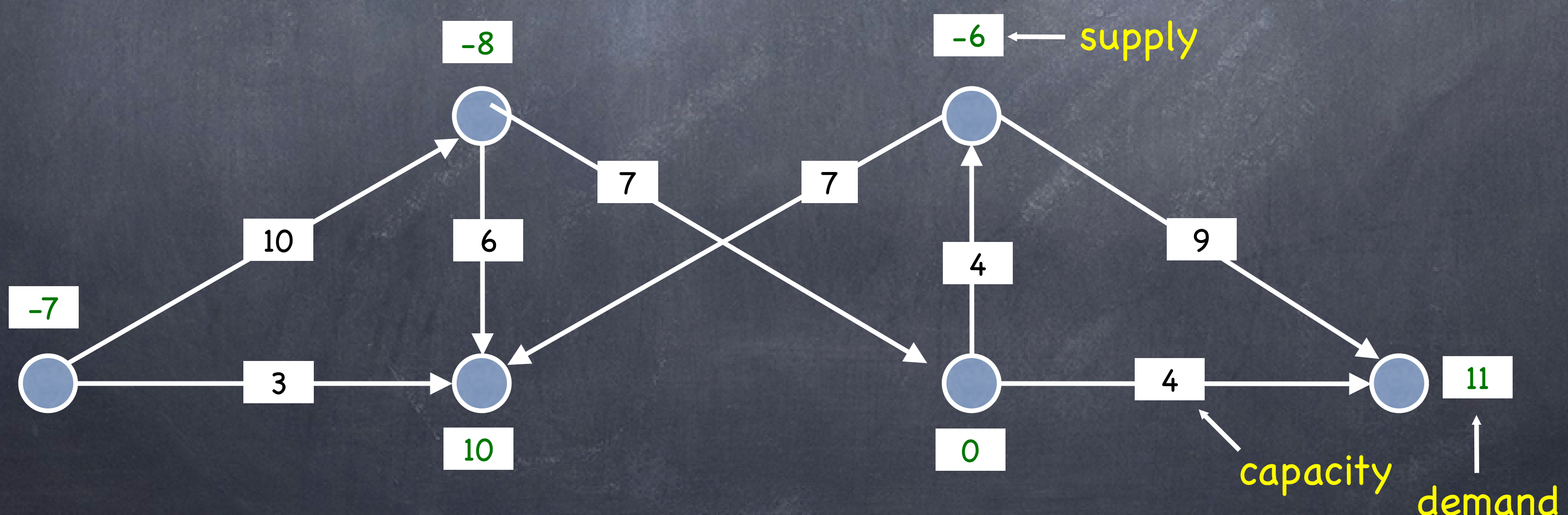


Circulation with Demands

- A feasible circulation satisfies:
  - Capacity condition: For each $e \in E$: $0 \leq f(e) \leq c(e)$
  - Conservation condition: For each $v \in V$:

    $$\sum_{e \text{ into } V} f(e) - \sum_{e \text{ out of } V} f(e) = d(v)$$

- Circulation problem: does there exist a feasible circulation?

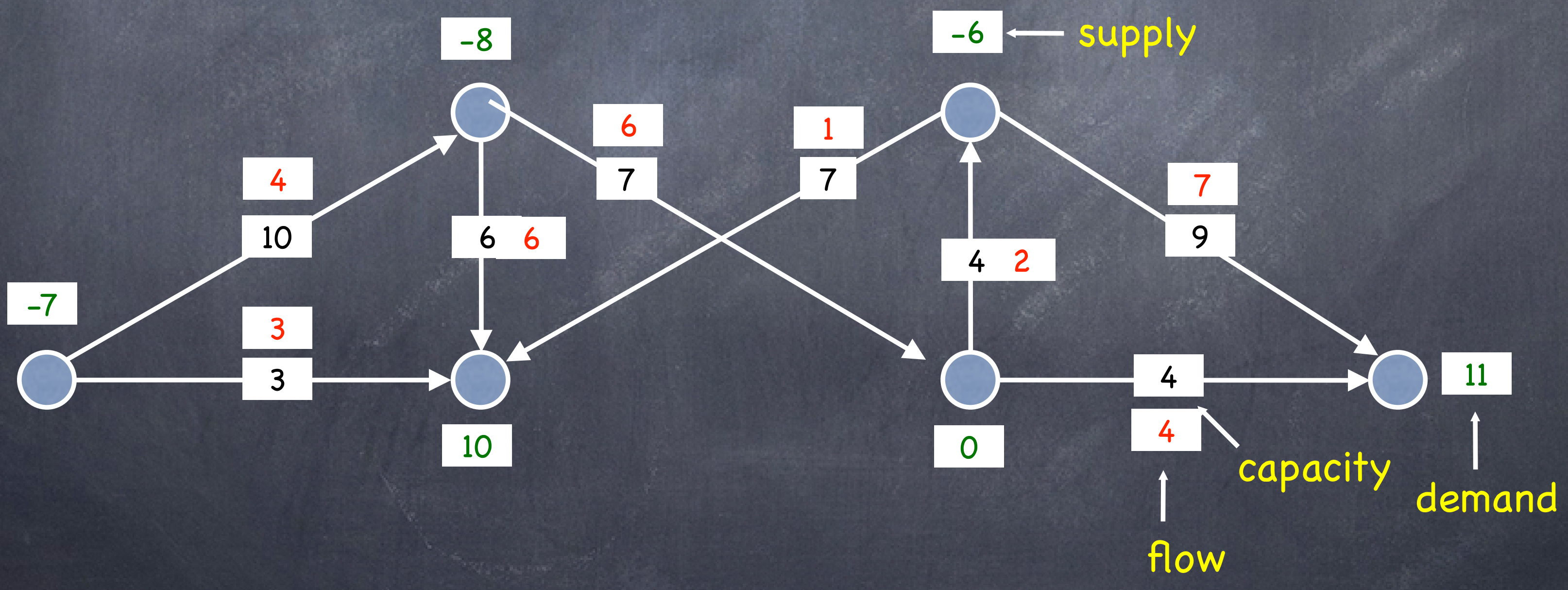


Note

- This is a flow network, but flow can enter and leave at many places.

Circulation with Demands

- Necessary condition: sum of supplies = sum of demands. $D = \sum_{v:\, d(v) > 0} d(v) = \sum_{v:\, d(v) < 0} -d(v)$

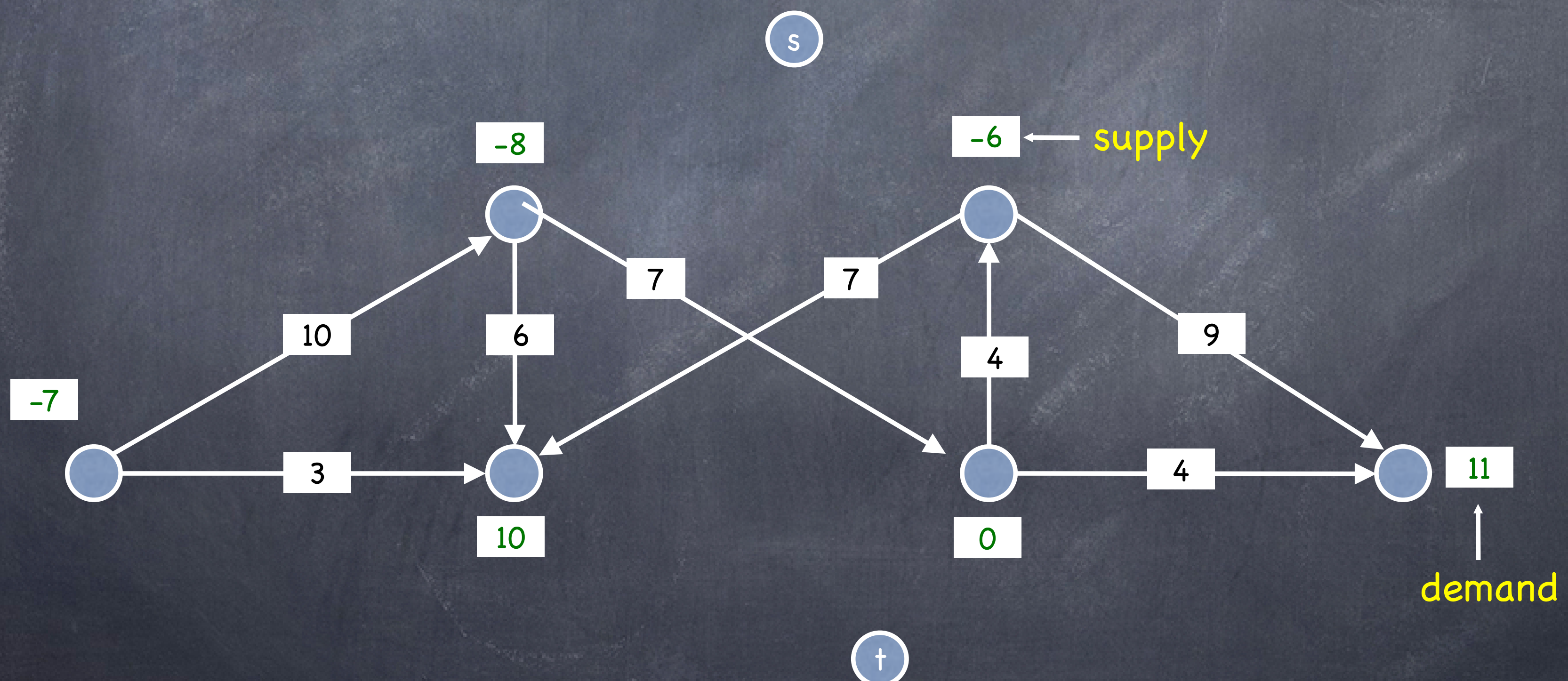


Exercise

- Can you solve the circulation problem using max flow?
- How to transform this into a flow network?
- How to use max. flow value to decide if there is a feasible circulation?

Max Flow Formulation

- Add new source s and sink t.
- For each supply node, add edge (s, v) with capacity -d(v).
- For each demand node, add edge (v, t) with capacity d(v).
- Claim: G has feasible circulation iff G' has max flow of value D.



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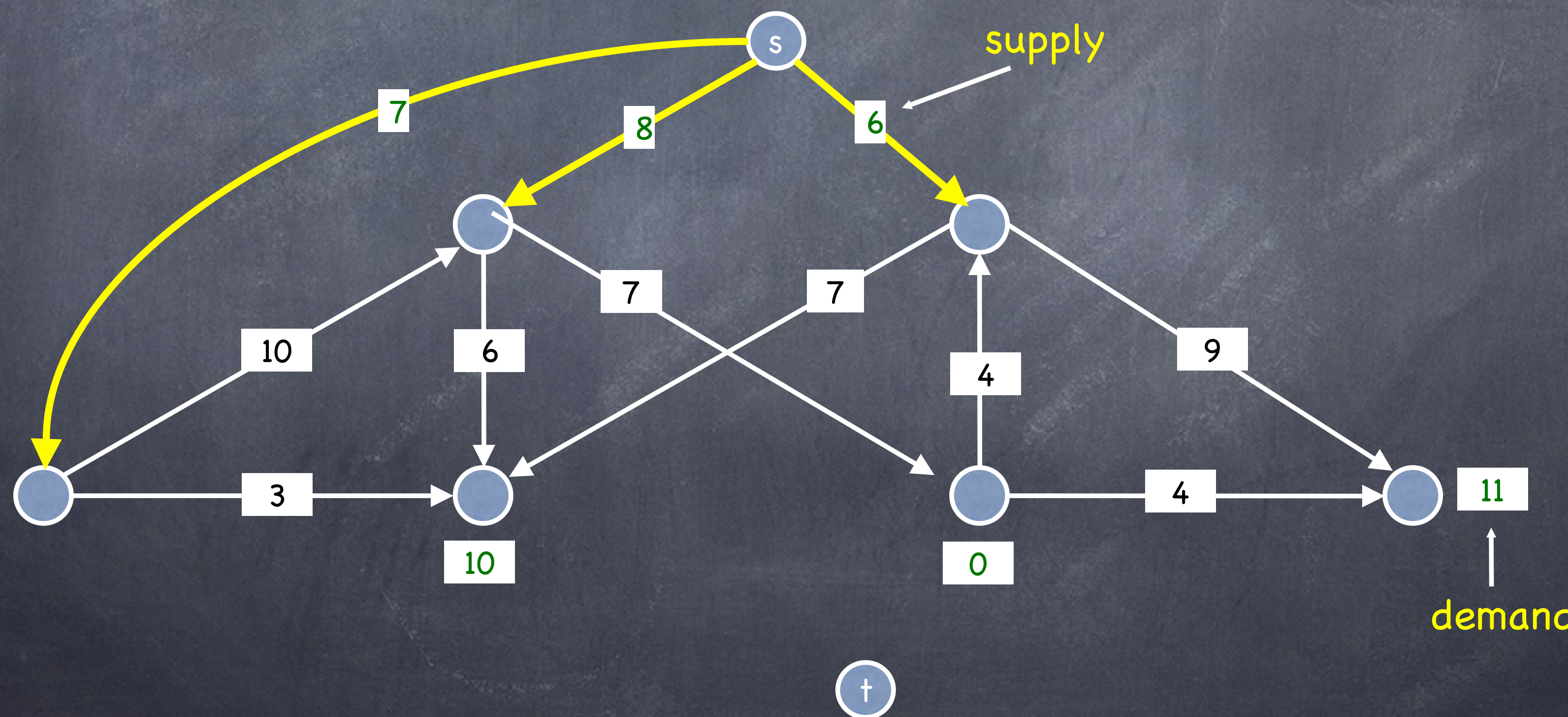

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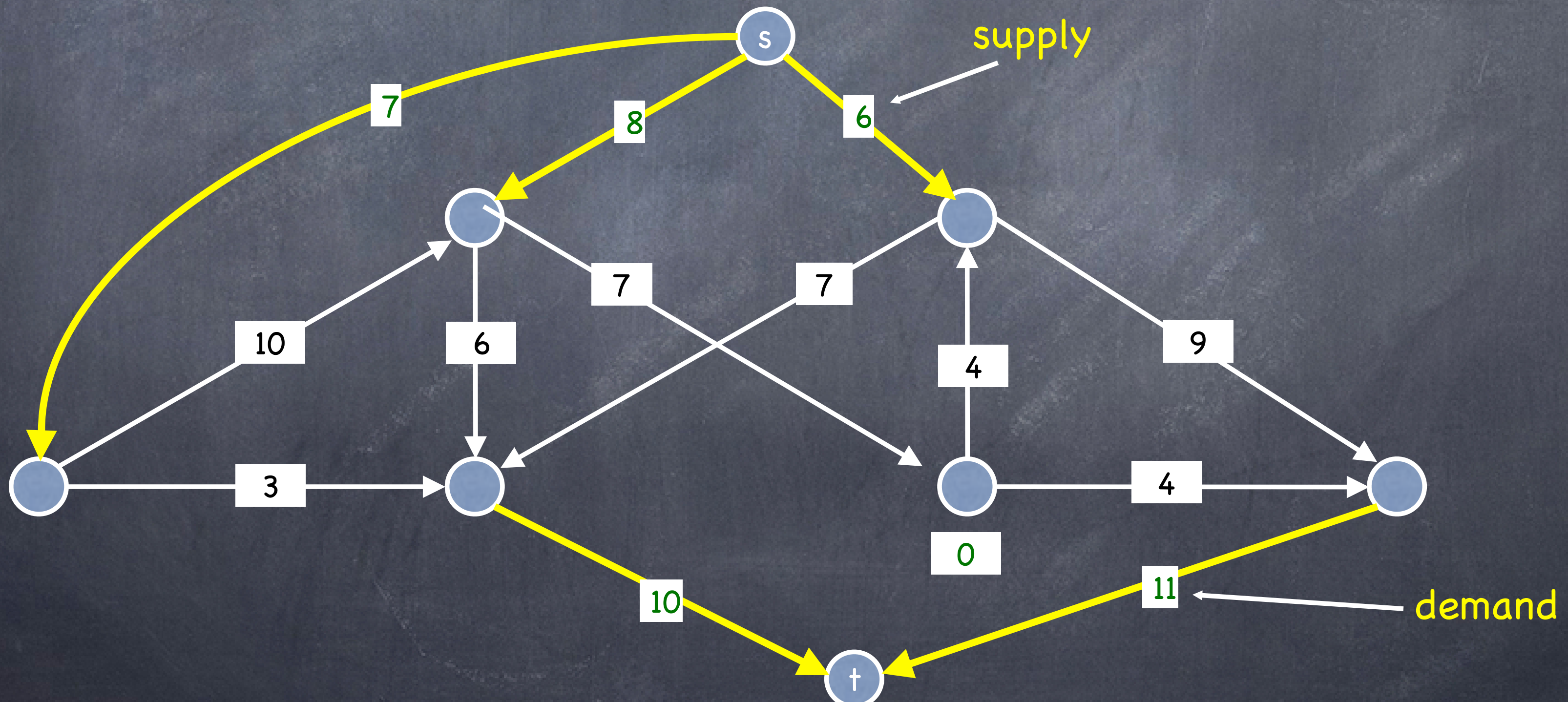


Max Flow Formulation

Claim: G has feasible circulation iff G' has max flow of value D.

Proof sketch

- Max flow in G′ is at most D (=total supply). Why?
- (→) Let f be a feasible circulation. Saturate all edges from source/sink to get a flow of value D.
- (←) Let f be a max flow of value D. Ignore flow on edges from source/sink to get a valid circulation.

Integrality

Corollary: if there is a feasible circulation, then there is one that is integer-valued (*)

(*) assuming capacities and demands are integers

Circulation with Demands and Lower Bounds

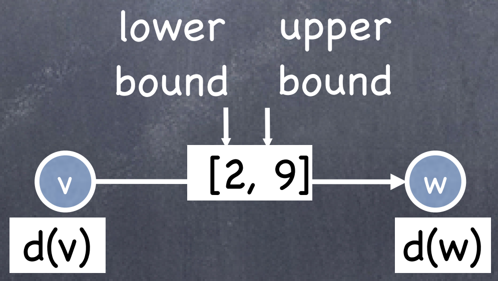
One more twist is very useful in applications. Require at least $\ell(e)$ units of flow on edge e

New capacity condition: $\ell(e) \leq f(e) \leq c(e), \quad e \in E$
Conservation condition: same

Problem: Does there exist a feasible circulation?

Circulation with Demands and Lower Bounds

This problem can be reduced to circulation problem without lower bounds in a couple different ways. (Further details omitted!)



Airline Scheduling

- Can you fulfill all of these flight segments with at most k planes?
  - Boston (6 AM) → DC (7 AM)
  - Philadelphia (7 AM) → Pittsburgh (8 AM)
  - DC (8 AM) → LA (11 AM)
  - Philadelphia (11 AM) → San Francisco (2 PM)
  - San Francisco (2:15 PM) → Seattle (3:15 PM)
  - Las Vegas (5 PM) → Seattle (6 PM)
- Assume 1 hour for maintenance

Airline Scheduling

Input: list of m flight segments (starting and ending airports and times)

Rules

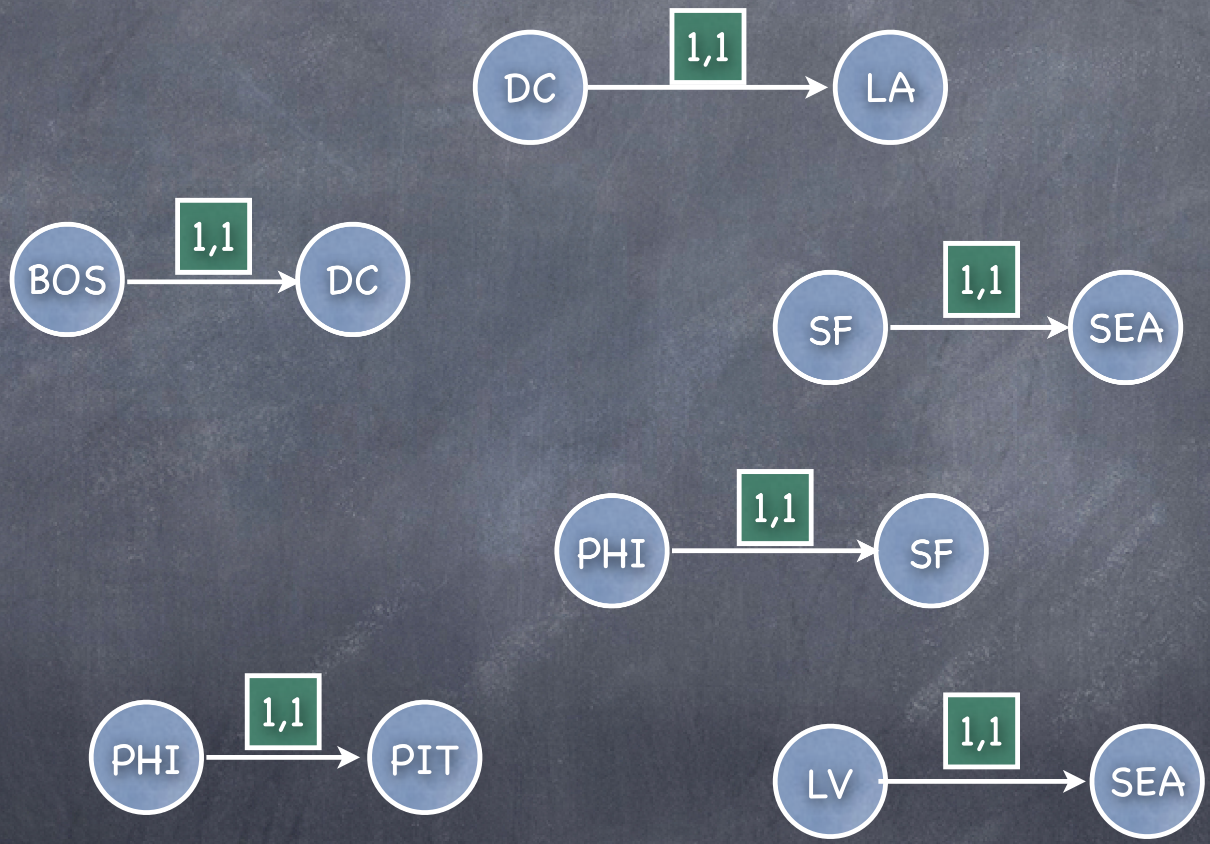
- One hour maintenance required before each flight
- It's ok to add a flight segment to get a plane to a different airport

Problem: Can you schedule all the flights with k planes?

Exercise

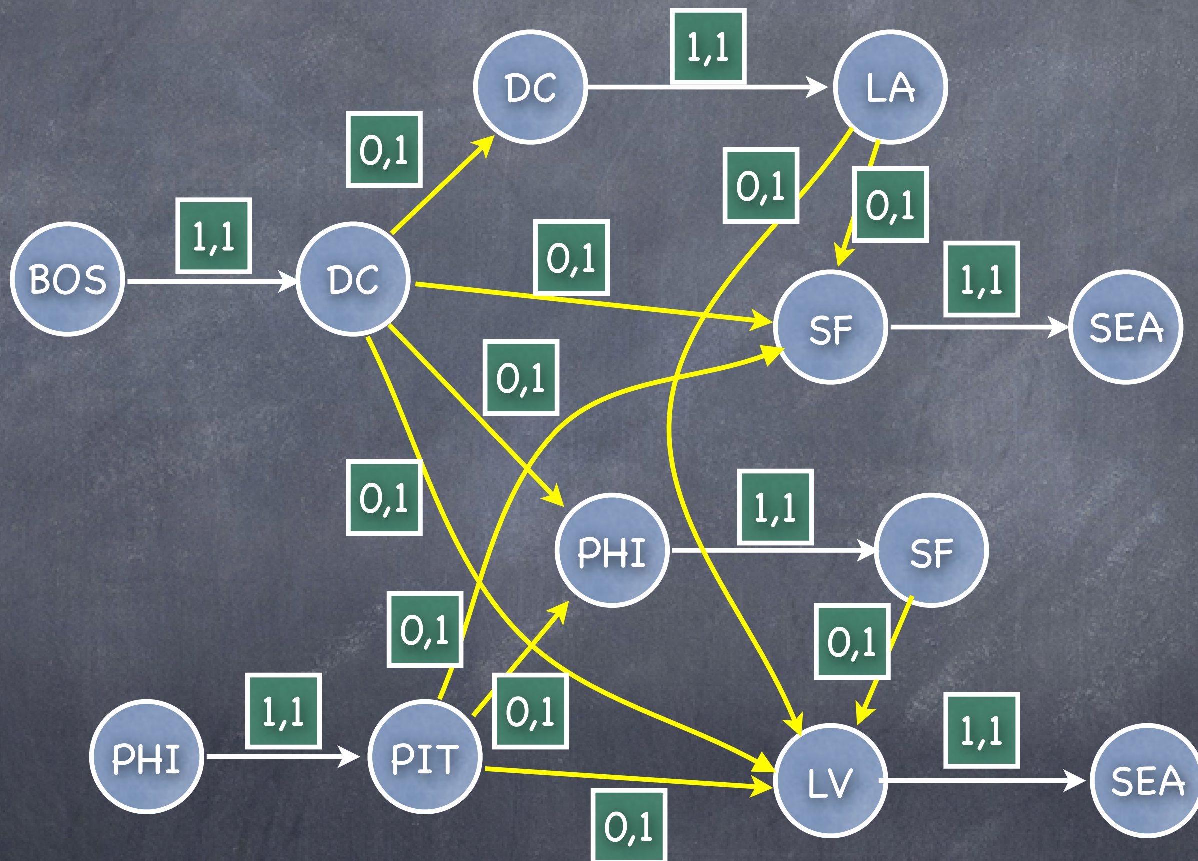
Solution

Add edge for each flight segment.



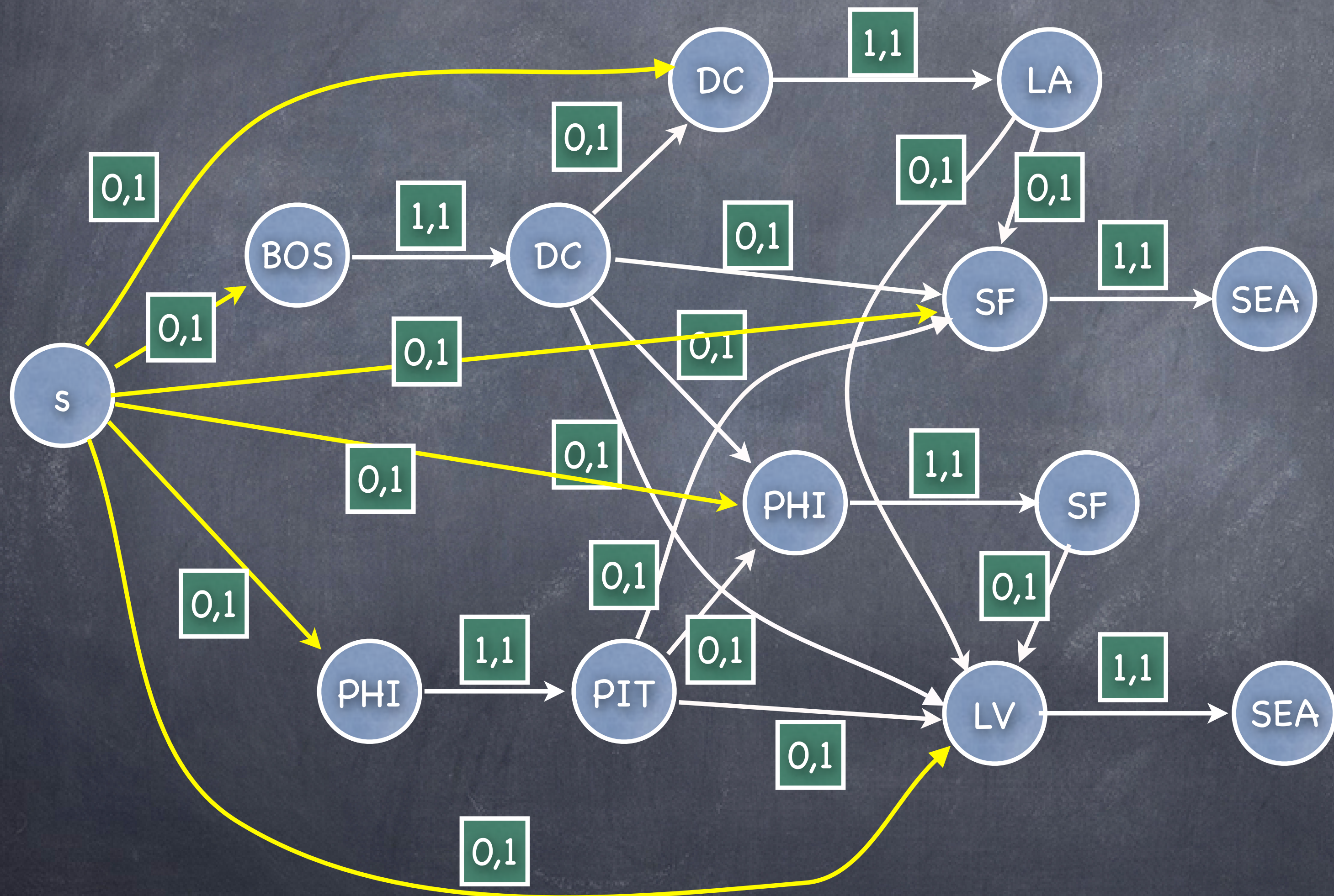
Solution

Add reachability edges.



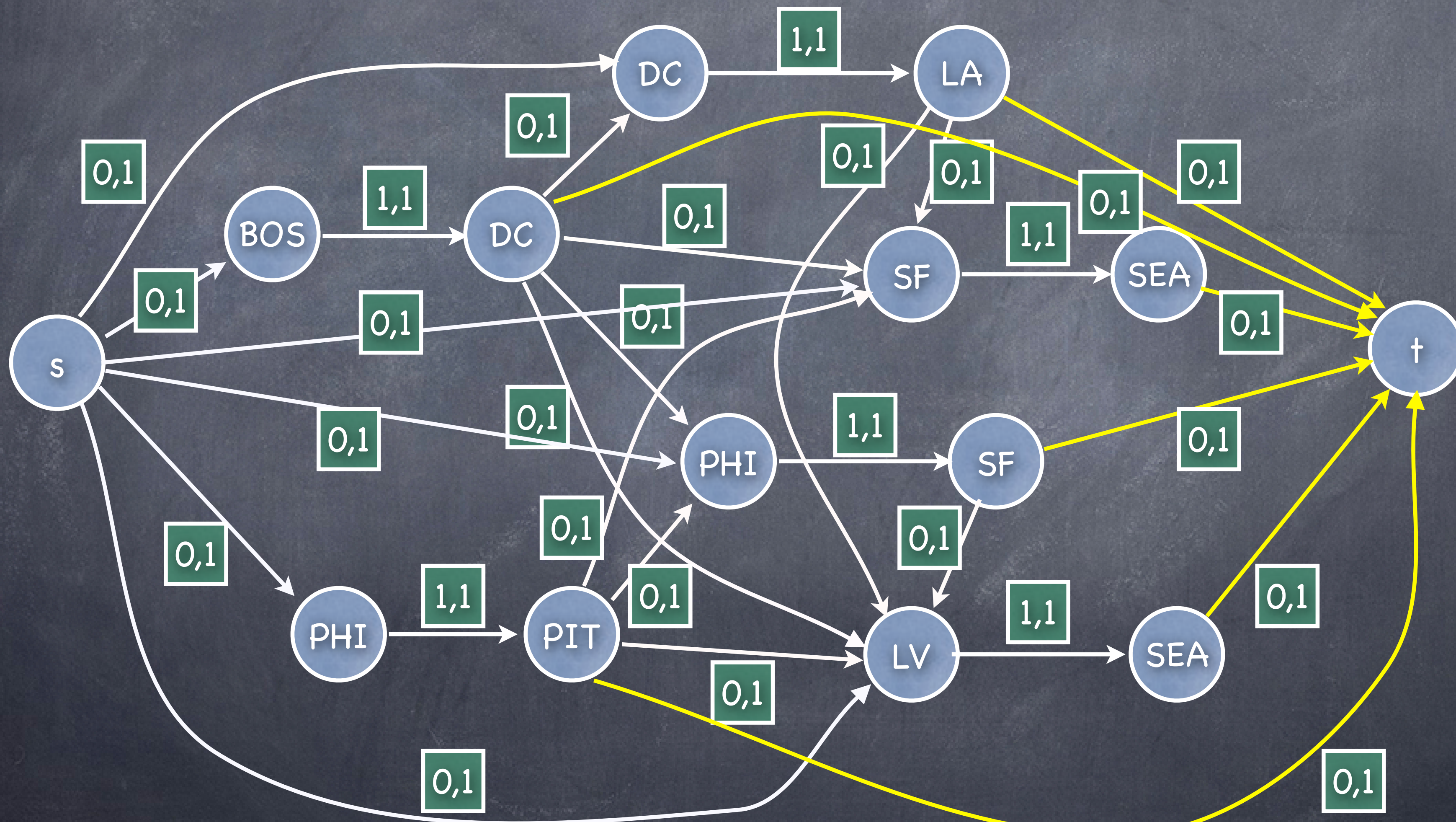
Solution

Add edges to model first flights.

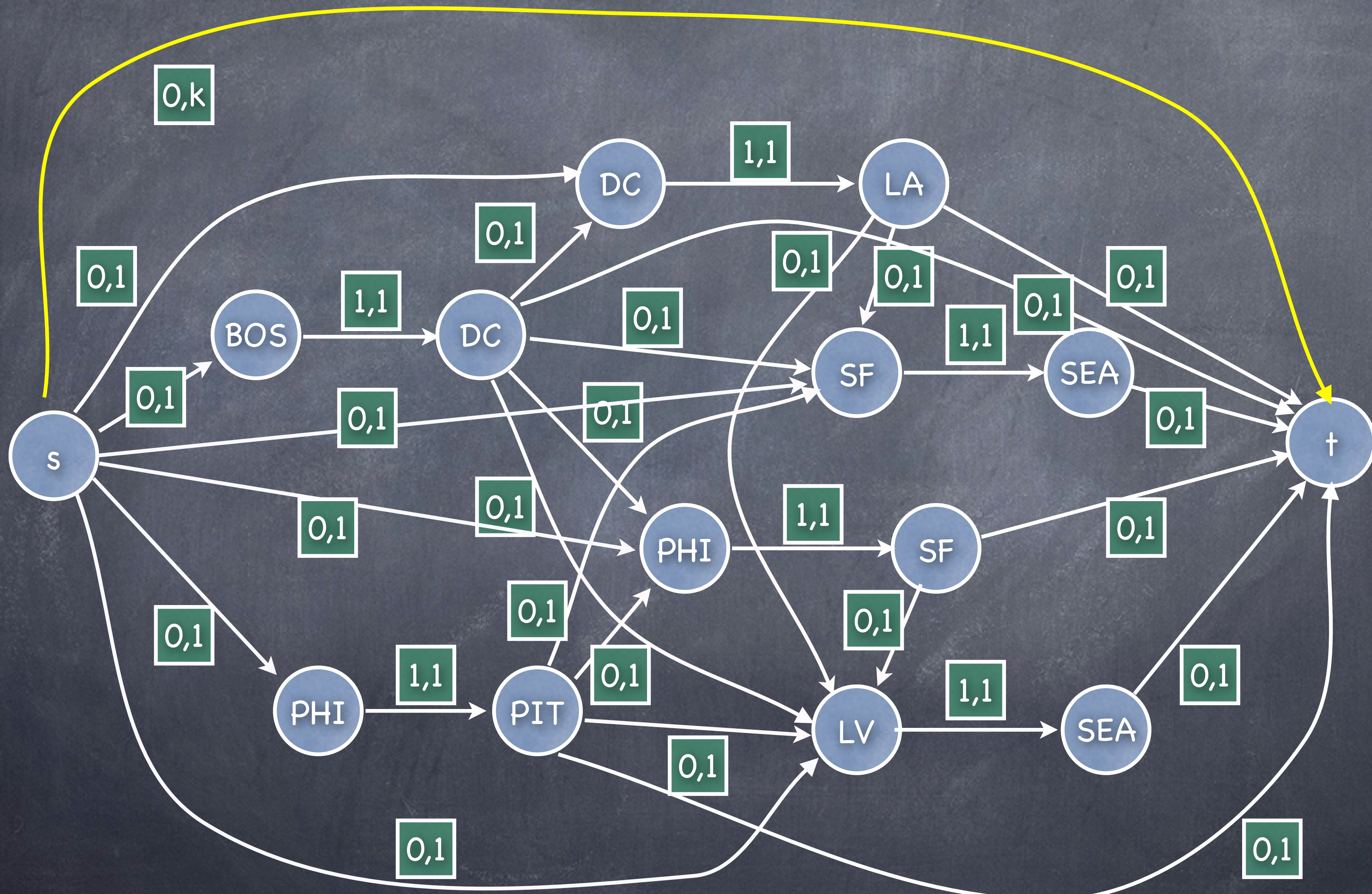


Solution

Add edges to model last flights.



Add edge for excess planes.



Add demands for all planes.